Math 279 Lecture 18 Notes

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October 26, 2021

1 Proving the Bounds in Hairer's Reconstruction Theorem

1.1 Recap: Constructing a candidate in Hairer's reconstruction theorem

Theorem 1.1 (Hairer's reconstruction theorem)**.** *Let F be γ-coherent. Then there exists a distribution $T = \mathcal{R}(F)$ such that*

$$|(T - F_x)(\psi_x^\delta)| \lesssim \begin{cases} \delta^\gamma & \gamma \neq 0, \\ \log \frac{1}{\delta} & \gamma = 0. \end{cases}$$

Here, the bound is uniform over $x \in K$, $\delta \in (0, 1]$, $\psi \in \mathcal{D}$ with $\|\psi\|_{C^r} \leq 1$.

We want to think of T as the value of some continuous operator $\mathcal{R}$. So far, we have a candidate for T when $\gamma > 0$. Here is an overview of what we have seen so far. We start from $\varphi \in \mathcal{D}$ such that $\int \varphi = 1$ and $\int \varphi x^k \, dx = 0$ for $0 < |k| < r$. From this φ, we constructed a suitable test function ρ of the form $\rho = \eta * \varphi$ with $\eta = \varphi^2$ and $\rho^{1/2} - \rho = \zeta * \varphi$, where $\zeta = \varphi^{1/2} - \varphi^2$. Recall that

$$\varphi^\delta(x) = \delta^{-d} \varphi(x/\delta), \qquad \varphi_a^\delta(x) = \delta^{-d} \varphi((x - a)/\delta), \qquad \varphi_a(x) = \varphi(x - a).$$

Observe that since $\int \zeta = 0$, we have $\int \zeta P \, dx = 0$ for any polynomial P of degree at most $r - 1$.

Here is the idea behind the construction of T: Indeed if we define convolution by

$$(T * \phi)(X) = T(\widetilde{\phi}_x), \qquad \widetilde{\phi}(z) = \phi(-z),$$

and if $\int \phi = 1$, it can be shown that $\lim_{\delta \to 0} T * \phi^\delta = T$. Recall that $\widehat{\rho}_x^n(y) = 2^{dn} \rho(2^n(y - x)) = \rho_x^{2^{-n}}(y)$, and since

$$\lim_{n \to \infty} T(\widehat{\rho}_x^n) = T,$$

from this we guess tat a good approximation for T satisfying the theorem is simply

$$T_n(x) = F_x(\widehat{\rho}_x^n).$$

Last time, we showed that indeed $T_n(x)$ converges when $\gamma > 0$, where convergence means that for any $\psi \in \mathcal{D}$, $\lim_n \langle T_n(x), \psi \rangle$ exists. The very form of ρ allows us to have the following representation:

$$T_n = T_1 + \sum_{k=1}^{n-1} (T_{k+1} - T_k),$$

where

$$T_{k+1}(x) - T_k(x) = F_x(\widehat{m}_x^k) = \int F_x(\widehat{\varphi}_y^k)\widehat{\zeta}_x^k(y)\,dy, \qquad m = \rho^{1/2} - \rho.$$

We can write this as

$$T_{k+1}(x) - T_k(x) = \underbrace{\int (F_x - F_y)(\widehat{\varphi}_y^k)\widehat{\xi}_x^k(y)\,dy}_{B_k} + \underbrace{\int F_y(\widehat{\varphi}_y^k)\widehat{\zeta}_x^k(y)\,dy}_{A_k}.$$

Last time, we showed that $\sum_k A_k$ converges no matter γ is. However, the bound for the second term is $|B_k| \lesssim 2^{-\gamma k}$, so for B_k, we get a pointwise bound that would imply the pointwise convergence only when $\gamma > 0$. In fact, when $\gamma \leq 0$, our candidate for T is $\lim_{n\to\infty} T_1 + \sum_{k=1}^{n-1} A_k = T_1 + \sum_{k=1}^{\infty} A_k$.

1.2 Proof of the bounds in the reconstruction theorem

We now try to prove that $(T - F_a)(\psi_a^\delta) \lesssim \delta^\gamma$. Here, a is fixed. We first focus on the case of $\gamma \leq 0$. Again, our T is the limit of $S_n = T_1 + \sum_{k=1}^{n-1} A_k$. To compare this with F_a, observe that

$$F_a = \lim_{n\to\infty} F_a(\widehat{\rho}_\cdot^n).$$

That is,

$$F_a(\psi) = \lim_{n\to\infty} \int F_a(\widehat{\rho}_x^n)\psi(x)\,dx.$$

In the same manner, we may write

$$F_a = G_1 + \sum_{k=1}^{\infty} (G_{k+1} - G_k), \qquad G_k(x) := F_a(\widehat{\rho}_x^k).$$

Also, we may find

$$\Gamma_n(x) = G_1(x) + \sum_{k=1}^{n-1} (G_{k+1}(x) - G_k(x)),$$

so that $\lim_{n\to\infty} \Gamma_n = F_a$. We wish to compare Γ_n to $S_n = T_1 + \sum_{k=1}^{n-1} A_k$. Observe that

$$C_k(x) = G_{k+1}(x) - G_k(x)$$

$$
\begin{aligned}
&= F_a(\widehat{\rho}_x^{k+1} - \widehat{\rho}_x^k) \\
&= F_a(\widehat{m}_x^k) \\
&= \int F_a(\widehat{\varphi}_y^k)\widehat{\zeta}_x^k(y)\,dy.
\end{aligned}
$$

We wish to estimate

$$
\begin{aligned}
|\langle A_k(x) - C_k(x), \psi_a^\delta\rangle| &= \left|\iint (F_y - F_a)(\widehat{\varphi}_y^k)\zeta_x^k(y)\,dy\,\psi_a^\delta(x)\,dx\right| \\
&= \left|\int (F_y - F_a)(\widehat{\varphi}_y^k)(\widehat{\zeta}^k * \psi_a^\delta)(y)\,dy\right| \\
&\lesssim \int 2^{k\tau}(|y-a| + 2^{-k})^{\gamma+\tau}|(\zeta^k * \psi_a^\delta)(y)|\,dy
\end{aligned}
$$

As a warmup, observe that we have the bound

$$
\le 2^{k\tau}(\delta + 2^{-k+1})^{\gamma+\tau}\|\psi^k * \psi_a^\delta\|_{L^1} \le 2^{k\tau}(\delta + 2^{-k+1})^{\gamma+\tau}\|\zeta\|_{L^1}\|\psi\|_{L^1}.
$$

Hence, if $\gamma < 0$,

$$
\begin{aligned}
\sum_{k:2^{-k}\ge\delta} |\langle A_k - C_k, \psi_a^\delta\rangle| &\lesssim \sum_{k:2^{-k}\ge\delta} 2^{-k\gamma} \\
&= \sum_{k\le|\log\delta|} 2^{-k\gamma} \\
&= \begin{cases} |\log\delta| & \gamma = 0 \\ (2^{-\gamma})^{|\log\delta|} = \delta^\gamma & \gamma < 0. \end{cases}
\end{aligned}
$$

Next, we concentrate on $\sum_{k:2^{-k}<\delta} |\langle A_k - C_k, \psi_a^\delta\rangle|$. To control this, we need a better estimate on $(\widehat{\zeta}^k * \psi_a^\delta)(y)$, which has a support contained in $B_a(2^{-k+\delta})$. Recall that $\int \zeta P = 0$ for any polynomial P of degree $< r$. Now

$$
\int \widehat{\zeta}^k(y)\psi_a^\delta(x)\,dx = \int \widehat{\zeta}_x^k(y)(\psi_a^\delta(x) - P_y(x))\,dx,
$$

where $P_y(x)$ is the Taylor polynomial of ψ_a^δ at y of degree $r-1$. Hence,

$$
\begin{aligned}
\left|\int \widehat{\zeta}^k(y)\psi_a^\delta(x)\,dx\right| &\lesssim \int |y-x|^r\|\psi_a^\delta\|_{C^r}\|\widehat{\xi}_x^k(y)|\,dx \\
&\lesssim 2^{-kr}\|\psi_a^\delta\|_{C^r} \\
&\lesssim 2^{-kr}\delta^{-d-r}\|\psi\|_{C^r}.
\end{aligned}
$$

Hence,

$$|\langle A_k - C_k, \psi_a^\delta\rangle| \lesssim 2^{k\tau}(\delta + 2^{-k+1})^{\gamma+\tau} 2^{-kr}\delta^{-d-r} \underbrace{(2^{-k}+\delta)^d}_{\text{volume of } B_a(2^{-k}+\delta)}.$$

Thus,

$$\sum_{k:2^{-k}\le\delta} |\langle A_k - C_k, \psi_a^\delta\rangle| \lesssim \sum_{k:2^{-k}\le\delta} 2^{-k(r-\tau)}\delta^{\gamma+\tau-r}$$

Provided $r > \tau$, we get

$$\begin{aligned} &\lesssim \delta^{r-\tau}\delta^{\gamma+\tau-r} \\ &= \delta^\gamma. \end{aligned}$$

This completes the proof when $\gamma \le 0$.

How about when $\gamma > 0$? We already know that the tail

$$\sum_{k:2^{-k}\le\delta} |\langle A_k - C_k, \psi_a^\delta\rangle| \lesssim \delta^\gamma.$$

We now argue that

$$\sum_{k:2^{-k}\le\delta} |\langle B_k, \psi_a^\delta\rangle| \lesssim \delta^\gamma$$

when $\gamma > 0$. Observe that

$$\begin{aligned} |\langle B_k, \psi_a^\delta\rangle| &= \left|\iint (F_x - F_y)(\widehat{\varphi}_y^k)\widehat{\zeta}_x^k(y)\,dy\,\psi_a^\delta(x)\,dx\right| \\ &\lesssim \left|\int 2^{k\tau}(|x-y| + 2^{-k})^{\gamma+\tau}\widehat{\zeta}_x^k(y)\,dy\,\psi_a^\delta(x)\,dx\right| \\ &\lesssim 2^{-k\gamma}\|\zeta\|_{L^1}\|\psi\|_{L^1}. \end{aligned}$$

Hence,

$$\sum_{k:2^{-k}\le\delta} |\langle B_k, \psi_a^\delta\rangle| \lesssim \sum_{k:2^{-k}\le\delta} 2^{-k\gamma} \lesssim \delta^\gamma.$$